Non-perturbative corrections to the planetary perturbation equation

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Abstract

This paper presents a systematic improvement in celestial dynamics theory by introducing a new symmetric form of particle dynamics equation. For open multi-body systems, the symmetric new equation can be applied to any translational reference frame, thus avoiding the need for inertial reference frame approximations and enhancing the accuracy of theoretical predictions. In the case of bound multi-body systems, applying the symmetric new equation allows for an extremely simplified derivation of the planetary perturbation equation in one step. Furthermore, by considering temporary thrust or impact forces acting on planets, or even considering any external forces acting on the bound system further to enhance the computational precision, a new correction equation is now established for the planetary perturbation that can be further imposed with non-perturbative interactions. This will assist in the prediction of the trajectory of asteroids affected by external forces and in the accurate calculation of the orbit of satellites.

Keywords: Celestial dynamics; Non-perturbative correction; Symmetric new equation; Planetary perturbation equation

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1. Introduction

The theoretical foundation of traditional celestial dynamics is Newton's second law, with the condition of an inertial reference frame. Based on this, it can rigorously derive a symmetrical form of the particle dynamics equation[1,2]. Assuming the existence of an inertial reference frame Σ in the universe, with a celestial body under consideration denoted as p and an actual celestial body chosen as a reference object o, we can, in principle, use Newton's second law to calculate the accelerations of these two bodies relative to the inertial reference frame Σ, which are respectively,

$$\begin{aligned} \boldsymbol{F}\big|_p &= m_p\,\boldsymbol{a}\big|_{p-\Sigma} \\ \boldsymbol{F}\big|_o &= m_o\,\boldsymbol{a}\big|_{o-\Sigma} \end{aligned} \tag{1}$$

To show the formal symmetry, the above equation can be written, again, as

$$\begin{aligned} \boldsymbol{a}\big|_{p-\Sigma} &= \frac{\boldsymbol{F}\big|_p}{m_p} \\ \boldsymbol{a}\big|_{o-\Sigma} &= \frac{\boldsymbol{F}\big|_o}{m_o} \end{aligned} \tag{2}$$

Obviously in the same inertial reference frame Σ, according to the Galilean transformation relationship[3] between relative acceleration and absolute acceleration, it can be obtained that

$$\boldsymbol{a}\big|_{p-\Sigma} - \boldsymbol{a}\big|_{o-\Sigma} = \left(\boldsymbol{a}\big|_{p-o}\right)_{\Sigma} \tag{3}$$

Now, with o as the reference origin, we establish a reference frame that is rotation-free with respect to the background of cosmic space, i.e., the translational reference frame labeled O (with the convention that the translational reference frame and its reference object are labeled by the corresponding uppercase and lowercase English letters respectively, in order to distinguish between them). Considering that the inertial reference frame is also by definition rotation-free with respect to the background of cosmic space, we have

$$\left(\boldsymbol{a}\big|_{p-o}\right)_{\Sigma} = \boldsymbol{a}\big|_{p-O} \tag{4}$$

We associate Eq.2, Eq.3, and Eq.4 to get

$$\frac{\boldsymbol{F}|_p}{m_p} - \frac{\boldsymbol{F}|_o}{m_o} = \boldsymbol{a}|_{p-O} \tag{5}$$

This is the recently introduced symmetric form of the particle dynamics equation (referred to as the symmetric new equation below). Its physical significance and multiple derivations have been discussed in detail in other articles[1,2]. This paper focuses on demonstrating its advantages in the application of celestial mechanics. It is evident that for open multi-body systems, the traditional approach of applying Newton's second law involves approximating an actual reference frame on the celestial bodies as an inertial frame. For example, when studying the relative motion of the Sun and Moon, the heliocentric reference frame is approximated as an inertial frame[4]. This approach, no matter how high the degree of approximation, is bound to introduce errors. To address this kind of errors just arising from the limited applicability of the theory itself, applying the symmetric new equation to directly solve celestial mechanics in translational reference frames simplifies the process, and eliminates errors by removing inertial reference frame approximations [2]. In traditional theory, a more precise approach also exists for solving the dynamics of a celestial body in a bound system (i.e. nearly isolated systems neglecting external forces). For instance, examining the gravitational perturbations on a planet within the entire solar system, there are well-known planetary perturbation differential equations[5-9]. Subsequently, this paper will re-examine the derivation and explore potential adjustments of the planetary perturbation differential equations using the symmetric new equation.

2. Validated derivation of the conventional planetary perturbation equation

Taking the example of a nearly isolated arbitrary multi-body system, the planetary perturbation equation can be rederived using the symmetric new equation. In the classical n-body dynamics of the solar system, where n is the Sun and the other n-1 celestial bodies are planets, the coordinates of the i-th planet $(i < n)$ in the heliocentric reference frame are directly introduced: (x_i, y_i, z_i). According to the symmetric new equation (Eq.5), the acceleration of any i-th planet relative to the heliocentric reference frame

(with the reference object as n, hence the corresponding translational reference frame is N) satisfies,

$$\frac{\boldsymbol{F}|_i}{m_i} - \frac{\boldsymbol{F}|_n}{m_n} = \boldsymbol{a}|_{i-N} \tag{6}$$

Decomposing this vector equation into three component equations in the (x,y,z) directions, we take the equation in the x direction for the next derivation

$$\ddot{x}_i = \frac{F_x|_i}{m_i} - \frac{F_x|_n}{m_n} \tag{7}$$

Assuming for the moment in this section that only gravitational interactions within the solar system are considered, then according to the law of gravity, we have

$$\begin{aligned} F_x|_i &= \sum_{j=1\ldots n}^{i\neq j} \frac{Gm_i m_j \left(x_j - x_i\right)}{\Delta_{ij}^3} \\ F_x|_n &= \sum_{j=1}^{n-1} \frac{Gm_n m_j \left(x_j - x_n\right)}{\Delta_{nj}^3} \end{aligned} \tag{8}$$

Simply substituting the specific calculation formula (Eq.8) for forces into the component symmetric new equation (Eq.7), immediately yields

$$\begin{aligned} \ddot{x}_i &= \sum_{j=1\ldots n}^{i\neq j} \frac{Gm_j \left(x_j - x_i\right)}{\Delta_{ij}^3} - \sum_{j=1}^{n-1} \frac{Gm_j \left(x_j - x_n\right)}{\Delta_{nj}^3} = \left[\sum_{j=1\ldots n-1}^{i\neq j} \frac{Gm_j \left(x_j - x_i\right)}{\Delta_{ij}^3} + \frac{Gm_n \left(x_n - x_i\right)}{\Delta_{in}^3}\right] - \sum_{j=1}^{n-1} \frac{Gm_j \left(x_j - x_n\right)}{\Delta_{nj}^3} \\ &= \sum_{j=1\ldots n-1}^{i\neq j} \frac{Gm_j \left(x_j - x_i\right)}{\Delta_{ij}^3} - \frac{G\left(m_n + m_j\right) x_i}{r_i^3} - \sum_{j=1\ldots n-1}^{i\neq j} \frac{Gm_j x_j}{r_j^3} \end{aligned} \tag{9}$$

As is customary, the perturbation function R_{ij} [5,8,9], which directly includes parameters from the third-party celestial bodies (all parameters with index j), is now introduced,

$$R_{ij} \equiv Gm_j \left(\frac{1}{\Delta_{ij}} - \frac{x_i x_j + y_i y_j + z_i z_j}{r_j^3} \right) \tag{10}$$

Substituting Eq.10 into the Eq.9 yields

$$\ddot{x}_i + G\frac{\left(m_i + m_n\right) x_i}{r_i^3} = \sum_{j=1\ldots n-1}^{i\neq j} \frac{\partial R_{ij}}{\partial x_i} \tag{11}$$

This is the fundamental differential equation for planetary perturbations in celestial mechanics[5-9]. The equations in the y and z directions can also be derived in a similar

manner to the equation above. It is worth noting that the traditional derivation of the planetary perturbation equation starts with the introduction of a center-of-mass reference frame, the application of Newton's second law in the center-of-mass reference frame, and then transforms to an actual reference frame (the heliocentric reference frame) employing coordinate transformations. Now, the same result is rigorously re-derived based on a completely different starting point. Therefore, the validity of the symmetric new equation (Eq.5) in the framework of classical mechanics is once again confirmed. The cancellation of the dependence on the inertial reference frame by the symmetric new equation allows for an extremely simplified derivation in one step, leading to a more concise and efficient application of the planetary perturbation equation.

3. Non-perturbative corrections to the planetary perturbation equation

In the previous section's derivation, it was actually assumed that every celestial body is subject to only gravitational interactions only with other objects within this n-body system. However, in reality, celestial bodies may be influenced by other non-gravitational interactions. More importantly, it should also be noted that the traditional derivation of the planetary perturbation equation introduces Newton's second law through the center of mass reference frame, appearing precise, but essentially approximating the inertial reference frame at a higher level by treating the center of mass of the whole celestial multi-body bound system under investigation as the origin of an inertial reference frame. Therefore, in certain instances, to achieve greater precision, it is necessary to consider the external forces acting on the n-body system from external sources. With the symmetric new equation (Eq.5), this problem can be solved rigorously and concisely.

Assuming that the i-th celestial body, subject to gravitational interactions within the multi-body system under investigation labeled $\boldsymbol{f}_{sg}\big|_i$, is also subject to the non-gravitational interactions labeled $\boldsymbol{f}_{ng}\big|_i$ and all kinds of interactions coming from the outside of the bound system uniformly labeled $\boldsymbol{f}_{os}\big|_i$, so that Eq.6 can be rewritten as

$$\frac{\left(\boldsymbol{f}_{sg}\big|_i+\boldsymbol{f}_{ng}\big|_i+\boldsymbol{f}_{os}\big|_i\right)}{m_i}-\frac{\left(\boldsymbol{f}_{sg}\big|_n+\boldsymbol{f}_{ng}\big|_n+\boldsymbol{f}_{os}\big|_n\right)}{m_n}=\boldsymbol{a}\big|_{i-N} \qquad (12)$$

Repeating the derivation from Eq.7 to Eq.11, it is not difficult to obtain that the dynamics in the x-direction satisfies

$$\ddot{x}_i = \sum_{j=1\ldots n-1}^{i\neq j} \frac{Gm_j\left(x_j - x_i\right)}{\Delta_{ij}^3} - \frac{G\left(m_n + m_j\right)x_i}{r_i^3} - \sum_{j=1\ldots n-1}^{i\neq j} \frac{Gm_j x_j}{r_j^3} + \left[\frac{\left(\boldsymbol{f}_{ng}\big|_i + \boldsymbol{f}_{os}\big|_i\right)}{m_i} - \frac{\left(\boldsymbol{f}_{ng}\big|_n + \boldsymbol{f}_{os}\big|_n\right)}{m_n}\right] \tag{13}$$

After the same introduction of the perturbation function R_{ij}, the modified planetary perturbation equation, after superimposing the non-perturbative forces $\boldsymbol{f}_{ng}$ and $\boldsymbol{f}_{os}$, can be expressed as:

$$\ddot{x}_i + G\frac{\left(m_i + m_n\right)x_i}{r_i^3} = \sum_{j=1\ldots n-1}^{i\neq j} \frac{\partial R_{ij}}{\partial x_i} + \left[\frac{\left(\boldsymbol{f}_{ng}\big|_i + \boldsymbol{f}_{os}\big|_i\right)}{m_i} - \frac{\left(\boldsymbol{f}_{ng}\big|_n + \boldsymbol{f}_{os}\big|_n\right)}{m_n}\right] \tag{14}$$

For non-gravitational interactions $\boldsymbol{f}_{ng}$, this can be the propulsive force of a rocket or the destructive force such as an explosion. Therefore, the above Eq.14 will provide some guidance in the precise calculation of satellite orbits and the theoretical analysis of the artificially altering trajectory of small asteroids.

4. Conclusion

This paper invokes the newly introduced symmetric form of the particle dynamics equation to investigate its theoretical significance in the improvement of astrodynamics. The symmetric new equation can be rigorously derived from Newton's second law on the basis of theoretically existing inertial reference frames, and this paper demonstrates that it has non-negligible advantages in the application to multi-body systems. For general open multi-body systems, computational accuracy can be improved by avoiding approximations of inertial reference frames. For the bound nearly isolated multi-body systems, the symmetric new equation can effectively simplify the derivation of the planetary perturbation equation. Based on this, the non-perturbative correction equation for planetary motion is obtained either by further considering the temporarily applied forces on the planets or by deeply investigating the forces coming from outside the bound system.

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